Homework Assignments

Bifurcations: Theory and Applications

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`http://dynamics.mi.fu-berlin.de/lectures/`

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Problem 25: Let Γ denote a topological group with a representation ρ on a Banach space X. Prove or disprove:

(i) ρ is strongly continuous if, and only if, $\lim_{\gamma_n \to e} \rho(\gamma_n)x = \rho(e)x$ for all $x \in X$.

(ii) ρ is strongly continuous if, and only if, $\lim_{\gamma_n \to e} \rho(\gamma_n) = \rho(e)$, in the operator norm of X.

Problem 26: Let Γ denote a topological group with a representation ρ on a Banach space X. Consider a bounded, linear projection $P : X \to U$ so that $X = U \oplus V$, where $Q := I - P : X \to V$. Prove that P and Q are Γ-equivariant if, and only if, U and V are Γ-invariant.

Problem 27: Let Γ denote a topological group with a representation ρ on $X = \mathbb{R}^N$, and for $f \in C^1$ consider the ODE

$$(1) \quad \dot{x}(t) = f(x(t)).$$

Prove that f is Γ-equivariant if, and only if, the following holds for all $\gamma \in \Gamma$:

$$t \mapsto x(t) \text{ is a solution } \iff t \mapsto \rho(\gamma)x(t) \text{ is a solution of (1).}$$

Problem 28: Consider the space $L^2(S^1)$, $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, of 2π-periodic, square-integrable, complex-valued funtions. The canonical representation of the group $SO(2) = S^1$ on $L^2(S^1)$ is given by

$$(\gamma f)(x) = f(\gamma + x).$$

Determine all irreducible complex subspaces of $L^2(S^1)$.

Hint: Consider the unitary Fourier transform $T : L^2(S^1) \to \ell_2(\mathbb{C})$ given by

$$a_k := \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} f(t) \mathrm{d}t$$

and the Fourier series

$$f(t) = \sum_{k \in \mathbb{Z}} a_k e^{ikt}.$$